Appendix: Tables and formulae

Multidimensional Fourier transform: integral relations

Space variable: $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$

Frequency variable: $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$

Relation	
Direct transform	$\hat{f}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle} \mathrm{d}x_1 \cdots \mathrm{d}x_d$
Inverse transform	$\mathcal{F}^{-1}\{\hat{f}\} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\omega}) e^{j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d \overset{\text{a.e.}}{=} f(\boldsymbol{x})$
Parseval	$\int_{\mathbb{R}^d} f(\boldsymbol{x}) g^*(\boldsymbol{x}) \mathrm{d}x_1 \cdots \mathrm{d}x_d = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\omega}) \hat{g}^*(\boldsymbol{\omega}) \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d$
Energy conservation	$\int_{\mathbb{R}^d} \lvert f(\boldsymbol{x})\rvert^2 \mathrm{d}x_1 \cdots \mathrm{d}x_d = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \lvert\hat{f}(\boldsymbol{\omega})\rvert^2 \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_d$
Integral	$\int_{\mathbb{R}^d} f(\boldsymbol{x}) \mathrm{d}x_1 \cdots \mathrm{d}x_d = \hat{f}(\boldsymbol{0})$
Moments	$\int_{\mathbb{R}^d} x_1^{n_1} \cdots x_d^{n_d} f(\boldsymbol{x}) \mathrm{d}x_1 \cdots \mathrm{d}x_d = j^{n_1 + \cdots + n_d} \left. \frac{\partial^{n_1} \cdots \partial^{n_d} \hat{f}(\boldsymbol{\omega})}{\partial\omega_1^{n_1} \cdots \partial\omega_d^{n_d}} \right\rvert_{\boldsymbol{\omega}=\boldsymbol{0}}$
Convolution: $(f * h)(\boldsymbol{x})$	$\int_{\boldsymbol{y} \in \mathbb{R}^d} f(\boldsymbol{y}) h(\boldsymbol{x} - \boldsymbol{y}) \mathrm{d}y_1 \cdots \mathrm{d}y_p \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \hat{f}(\boldsymbol{\omega}) \cdot \hat{h}(\boldsymbol{\omega})$

Multidimensional Fourier transform: properties

Operation	$f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d$	$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle} \mathrm{d}x_1 \cdots \mathrm{d}x_d$
Linearity	$a_1 f_1(\boldsymbol{x}) + a_2 f_2(\boldsymbol{x})$	$a_1 \hat{f}_1(\boldsymbol{\omega}) + a_2 \hat{f}_2(\boldsymbol{\omega})$
Separability	$f(\boldsymbol{x}) = \prod_{i=1}^{d} f_i(x_i)$	$\hat{f}(\boldsymbol{\omega}) = \prod_{i=1}^{d} \hat{f}_i(\omega_i)$
Duality	$\hat{f}(\boldsymbol{x})$	$(2\pi)^d f(-\boldsymbol{\omega})$
Real-valued signal	$f(\boldsymbol{x})$ real	$\left(\hat{f}(\boldsymbol{\omega})\right)^* = \hat{f}(-\boldsymbol{\omega})$ (Hermitian symmetry)
Reflection	$f(-\boldsymbol{x})$	$\hat{f}(-\boldsymbol{\omega})$
Shift	$f(\boldsymbol{x} - \boldsymbol{x}_0)$	$e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}_0 \rangle} \hat{f}(\boldsymbol{\omega})$
Modulation	$e^{j\langle \boldsymbol{\omega}_0, \boldsymbol{x} \rangle} f(\boldsymbol{x})$	$\hat{f}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
Scaling	$f(\boldsymbol{x}/a)$	$\|a\|^d \hat{f}(a\boldsymbol{\omega})$
Affine transformation	$f(\boldsymbol{A}\boldsymbol{x})$	$\|\det \boldsymbol{A}\|^{-1} \hat{f}\left((\boldsymbol{A}^{-1})^T \boldsymbol{\omega}\right)$
Convolution	$(f * g)(\boldsymbol{x})$	$\hat{f}(\boldsymbol{\omega}) \cdot \hat{g}(\boldsymbol{\omega})$
Differentiation	$\frac{\partial^n f(\boldsymbol{x})}{\partial x_i^n}$	$(j\omega_i)^n \hat{f}(\boldsymbol{\omega})$
Multiplication	$f(\boldsymbol{x}) \cdot g(\boldsymbol{x})$	$\frac{1}{(2\pi)^d} (\hat{f} * \hat{g})(\boldsymbol{\omega})$
Multiplication by monomial	$x_i^n f(\boldsymbol{x})$	$j^n \frac{\partial^n \hat{f}(\boldsymbol{\omega})}{\partial \omega_i^n}$

Useful 1D Fourier-transform pairs

$f(x), \quad x \in \mathbb{R}$	$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-j\omega x} \mathrm{d}x$
Finite-energy functions	
$\mathrm{rect}(x) = \beta^0(x) = \begin{cases} 1, & \lvert x\rvert \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$	$\mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{(\omega/2)}$
$\mathrm{tri}(x) = \beta^1(x) = \begin{cases} 1 - \lvert x\rvert, & \lvert x\rvert \le 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	$\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2$
$\beta^n(x) = \left(\beta^0 * \beta^{n-1}\right)(x)$	$\mathrm{sinc}^{n+1}\left(\frac{\omega}{2\pi}\right)$
$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	$e^{-\omega^2/2}$
$e^{-a\lvert x\rvert}$ with $a > 0$	$\frac{2a}{a^2 + \omega^2}$
Generalized functions in $\mathcal{S}'$	
$\delta(x)$	1
$\delta(x - x_0)$	$e^{-j\omega x_0}$
$\sum_{k\in\mathbb{Z}} \delta(x - k)$	$2\pi \sum_{n\in\mathbb{Z}} \delta(\omega - 2\pi n)$
1	$(2\pi) \cdot \delta(\omega)$
$e^{j\omega_0 x}$	$(2\pi) \cdot \delta(\omega - \omega_0)$
$\cos(\omega_0 x)$	$\pi\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right]$
$\sin(\omega_0 x)$	$j\pi\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]$

Multidimensional Fourier-transform pairs

$f(\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$	$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle} \mathrm{d}x_1 \cdots \mathrm{d}x_d$
Finite-energy functions	
$f(\boldsymbol{x}) = \prod_{i=1}^{d} f_i(x_i)$	$\prod_{i=1}^{d} \hat{f}_i(\omega_i)$ with $\hat{f}_i = \mathcal{F}_{\mathrm{1D}}\{f_i\}$
$\mathrm{rect}(\boldsymbol{x}) \triangleq \prod_{i=1}^{d} \mathrm{rect}(x_i)$	$\mathrm{sinc}\left(\frac{\boldsymbol{\omega}}{2\pi}\right) \triangleq \prod_{i=1}^{d} \mathrm{sinc}\left(\frac{\omega_i}{2\pi}\right)$
$\mathrm{circle}(x_1, x_2) = \begin{cases} 1, & \sqrt{x_1^2 + x_2^2} \le \frac{1}{2} \\ 0, & \text{otherwise (circle)} \end{cases}$	$2\pi \frac{J_1\left(\sqrt{\omega_1^2 + \omega_2^2}\right)}{\sqrt{\omega_1^2 + \omega_2^2}}$ ($J_1(x)$: Bessel first kind)
$\mathrm{sinc}(\boldsymbol{x})$	$\mathrm{rect}\left(\frac{\boldsymbol{\omega}}{2\pi}\right)$
$(2\pi)^{-d/2}\, e^{-\lVert \boldsymbol{x} \rVert^2/2}, \quad \boldsymbol{x} \in \mathbb{R}^d$	$e^{-\lVert \boldsymbol{\omega} \rVert^2/2}$
Generalized functions in $\mathcal{S}'(\mathbb{R}^d)$	
$\delta(\boldsymbol{x}) = \prod_{i=1}^{d} \delta(x_i)$	1
$\delta(\boldsymbol{x} - \boldsymbol{x}_0)$	$e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}_0 \rangle}$
$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \delta(\boldsymbol{x} - \boldsymbol{k})$	$(2\pi)^d \sum_{\boldsymbol{n} \in \mathbb{Z}^d} \delta(\boldsymbol{\omega} - 2\pi \boldsymbol{n})$
1	$(2\pi)^d \cdot \delta(\boldsymbol{\omega})$
$e^{j\langle \boldsymbol{\omega}_0, \boldsymbol{x} \rangle}$	$(2\pi)^d \cdot \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
$\frac{1}{\lVert \boldsymbol{x} \rVert}, \quad \boldsymbol{x} \in \mathbb{R}^d,\ d > 1$	$\frac{c_d}{\lVert \boldsymbol{\omega} \rVert^{d-1}}$ with $c_d = (4\pi)^{\frac{d-1}{2}}\, \Gamma(\frac{d-1}{2})$ $[c_2 = 2\pi]$

Definition: $\mathcal{F}\{f\} = \hat{f} \in \mathcal{S}'(\mathbb{R}^d)$ is the generalized Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ iff $\langle f, \phi \rangle = \frac{1}{(2\pi)^d} \langle \hat{f}, \hat{\phi} \rangle$, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz' class of test functions).

Radial functions: Fourier-transform pairs

$f(\boldsymbol{x}) = \mathcal{F}^{-1}\{\hat{f}\}(\boldsymbol{x})$	$\hat{f}(\boldsymbol{\omega}) = \mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle} \mathrm{d}x_1 \cdots \mathrm{d}x_d$
Radial function	
$f(\boldsymbol{x}) = f(\\|\boldsymbol{x}\\|), \quad \\|\boldsymbol{x}\\| = \sqrt{x_1^2 + \cdots + x_d^2}$	$\hat{f}(\boldsymbol{\omega}) = \hat{f}(\\|\boldsymbol{\omega}\\|), \quad \\|\boldsymbol{\omega}\\| = \sqrt{\omega_1^2 + \cdots + \omega_d^2}$
Gaussian	
$e^{-\\|\boldsymbol{x}\\|^2/2}, \quad \boldsymbol{x} \in \mathbb{R}^d$	$(2\pi)^{d/2} e^{-\\|\boldsymbol{\omega}\\|^2/2}$
Thin-plate splines	
$\\|\boldsymbol{x}\\|^s, \quad \boldsymbol{x} \in \mathbb{R}^d, s \in \mathbb{C}, -(s+d), s \notin 2\mathbb{N}$	$\frac{c_{s,d}}{\\|\boldsymbol{\omega}\\|^{s+d}}$ with $c_{s,d} = \frac{2^{s+d}\pi^{\frac{d}{2}}\Gamma(\frac{s+d}{2})}{\Gamma(-\frac{s}{2})}$
$\\|\boldsymbol{x}\\|^{2n} \log \\|\boldsymbol{x}\\|, \quad \boldsymbol{x} \in \mathbb{R}^d, n \in \mathbb{N}$	$\frac{c'_{n,d}}{\\|\boldsymbol{\omega}\\|^{2n+d}}$ with $c'_{n,d} = (-1)^{n+1} 2^{2n-1+d} \pi^{\frac{d}{2}} \Gamma(n+\frac{d}{2})\, n!$
Hyperquadrics	
$(a^2 + \\|\boldsymbol{x}\\|^2)^s, \quad a > 0, s \in \mathbb{C}\backslash\{0\}, \boldsymbol{x} \in \mathbb{R}^d$	$\frac{2^{s+1}(2\pi)^{\frac{d}{2}}}{\Gamma(-s)} \left(\frac{a}{\\|\boldsymbol{\omega}\\|}\right)^{\frac{d}{2}+s} K_{\frac{d}{2}+s}(a\\|\boldsymbol{\omega}\\|)$
	$K_\nu(x) \geq 0$: Modified Bessel function of second kind
$\\|\boldsymbol{x}\\|^\nu K_\nu(\\|\boldsymbol{x}\\|), \quad \boldsymbol{x} \in \mathbb{R}^d, \nu \in \mathbb{R}^+$	$\frac{b_{\nu,d}}{(1 + \\|\boldsymbol{\omega}\\|^2)^{\nu+\frac{d}{2}}}$ with $b_{\nu,d} = 2^{\nu+d-1}\pi^{\frac{d}{2}}\Gamma(\nu + \frac{d}{2})$

Multidimensional z-transform: properties

Operation	Discrete signal	z-transform with $\boldsymbol{z} = (z_1, \dots, z_d)$
Definition	$x[\boldsymbol{k}], \quad \boldsymbol{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$	$X(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} x[\boldsymbol{k}]\ z_1^{-k_1} \cdots z_d^{-k_d}$
Linearity	$a_1 x_1[\boldsymbol{k}] + a_2 x_2[\boldsymbol{k}]$	$a_1 X_1(\boldsymbol{z}) + a_2 X_2(\boldsymbol{z})$
Separability	$x[\boldsymbol{k}] = x_1[k_1] \times \cdots \times x_d[k_d]$	$X(\boldsymbol{z}) = X_1(z_1) \times \cdots \times X_d(z_d)$
Delay	$x[\boldsymbol{k} - \boldsymbol{k}_0]$	$\boldsymbol{z}^{-\boldsymbol{k}_0} X(\boldsymbol{z}) = z_1^{-k_{01}} \cdots z_d^{-k_{0d}} X(\boldsymbol{z})$
Reflection	$x^T[\boldsymbol{k}] = x[-\boldsymbol{k}]$	$X(z_1^{-1}, \dots, z_d^{-1})$
Convolution	$(h * x)[\boldsymbol{k}] = \sum_{\boldsymbol{k}_1 \in \mathbb{Z}^d} h[\boldsymbol{k}_1] x[\boldsymbol{k} - \boldsymbol{k}_1]$	$Y(\boldsymbol{z}) = H(\boldsymbol{z}) \cdot X(\boldsymbol{z})$

1D z-transform pairs

$a[k]$	$X(z) = \sum_{k \in \mathbb{Z}} a[k]\ z^{-k}$	ROC
Unit (or Kronecker) impulse at $k = k_0$		
$\delta[k - k_0] = \begin{cases} 1, & k = k_0 \\ 0, & \text{otherwise} \end{cases}$	z^{-k_0}	$z \in \mathbb{C} \backslash \{0\}$
Unit step		
$u[k] = \begin{cases} 1, & k \geq 0 \\ 0, & \text{otherwise} \end{cases}$	$\sum_{k=0}^{+\infty} z^{-k} = \frac{1}{1 - z^{-1}}$	$\|z\| > 1$
Rectangular pulse of size m		
$u_m[k] = \begin{cases} 1, & 0 \leq k < m \\ 0, & \text{otherwise} \end{cases}$	$\sum_{k=0}^{m-1} z^{-k} = \frac{1 - z^{-m}}{1 - z^{-1}}$	$z \in \mathbb{C} \backslash \{0\}$
Exponentials		
$a^k \cdot u[k]$ (causal)	$\frac{1}{1 - az^{-1}}$	$\|a\| < \|z\|$
$a^{\|k\|}$ (symmetric)	$\frac{1 - a^2}{(1 - az^{-1})(1 - az)}$	$\|a\| < \|z\| < \|a\|^{-1}$
Modulated exponentials (causal)		
$r^k \cos(\omega_0 k) \cdot u[k]$	$\frac{1 - (r \cos \omega_0) z^{-1}}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}}$	$\|r\| < \|z\|$
$r^k \sin(\omega_0 k) \cdot u[k]$	$\frac{1 - (r \sin \omega_0) z^{-1}}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}}$	$\|r\| < \|z\|$